

## SOME DIFFERENCE SEQUENCE SPACES OF INFINITE MATRIX AND ORLICZ FUNCTION

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**ABSTRACT.** In this paper we introduce some generalized difference sequence spaces of ideal convergence, infinite matrix and sequence of Orlicz functions of order  $\alpha$  over  $n$ -normed spaces. We study some basic topological and algebraic properties of these spaces. We also investigate some inclusion relations related to these spaces.

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### 1. INTRODUCTION AND PRELIMINARIES

The concept of 2-normed spaces was initially developed by Gähler [13] in the mid of 1960's, while that of  $n$ -normed spaces one can see in Misiak [25]. Since then, many others have studied this concept and obtained various results, see Gunawan ([14], [15]) and Gunawan and Mashadi [16] and many others. Let  $n \in \mathbb{N}$  and  $X$  be a linear space over the real field  $\mathbb{R}$  of dimension  $d$ , where  $d \geq n \geq 2$ . A real valued function  $\|\cdot, \dots, \cdot\|$  on  $X^n$  satisfying the following four conditions:

- (1)  $\|x_1, x_2, \dots, x_n\| = 0$  if and only if  $x_1, x_2, \dots, x_n$  are linearly dependent in  $X$ ;
- (2)  $\|x_1, x_2, \dots, x_n\|$  is invariant under permutation;
- (3)  $\|\lambda x_1, x_2, \dots, x_n\| = |\lambda| \|x_1, x_2, \dots, x_n\|$  for any  $\lambda \in \mathbb{R}$ , and
- (4)  $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$

is called an  $n$ -norm on  $X$ , and the pair  $(X, \|\cdot, \dots, \cdot\|)$  is called an  $n$ -normed space over the field  $\mathbb{R}$ .

For example, we may take  $X = \mathbb{R}^n$  being equipped with the  $n$ -norm  $\|x_1, x_2, \dots, x_n\|_E =$  the volume of the  $n$ -dimensional parallelepiped spanned by the vectors  $x_1, x_2, \dots, x_n$  which may be given explicitly by the formula

$$\|x_1, x_2, \dots, x_n\|_E = |\det(x_{ij})|,$$

where  $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$  for each  $i = 1, 2, \dots, n$ , where script  $E$  denotes Euclidean space. Let  $(X, \|\cdot, \dots, \cdot\|)$  be an  $n$ -normed space of dimension  $d \geq n \geq 2$  and  $\{a_1, a_2, \dots, a_n\}$  be linearly independent set in  $X$ . Then the following function  $\|\cdot, \dots, \cdot\|_\infty$  on  $X^{n-1}$  defined by

$$\|x_1, x_2, \dots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \dots, x_{n-1}, a_i\| : i = 1, 2, \dots, n\}$$

defines an  $(n-1)$ -norm on  $X$  with respect to  $\{a_1, a_2, \dots, a_n\}$ .

A sequence  $(x_k)$  in an  $n$ -normed space  $(X, \|\cdot, \dots, \cdot\|)$  is said to converge to

some  $L \in X$  if

$$\lim_{k \rightarrow \infty} \|x_k - L, z_1, \dots, z_{n-1}\| = 0 \text{ for every } z_1, \dots, z_{n-1} \in X.$$

A sequence  $(x_k)$  in an  $n$ -normed space  $(X, \|\cdot, \dots, \cdot\|)$  is said to be Cauchy if

$$\lim_{k, i \rightarrow \infty} \|x_k - x_i, z_1, \dots, z_{n-1}\| = 0 \text{ for every } z_1, \dots, z_{n-1} \in X.$$

If every Cauchy sequence in  $X$  converges to some  $L \in X$ , then  $X$  is said to be complete with respect to the  $n$ -norm. Any complete  $n$ -normed space is said to be  $n$ -Banach space.

The notions of statistical convergence and convergence in density for sequences has been in the literature, under different guises, since the early part of the last century. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Statistical convergence was recently investigated by Fast [12] and Schoenberg [35] independently.

The concept of ideal convergence was first introduced by P. Kostyrko et al. [21] as a generalization of statistical convergence which was further studied in topological spaces by Das et al. [3]. More applications of ideals can be seen in ([3], [4]). We continue in this direction and introduce  $I$ -convergence of generalized sequences in more general setting.

A family  $I \subseteq 2^X$  of subsets of a non empty set  $X$  is said to be an ideal in  $X$  if

- (1)  $\phi \in I$ ;
- (2)  $A, B \in I$  imply  $A \cup B \in I$ ;
- (3)  $A \in \mathcal{I}, B \subseteq A$  imply  $B \in I$ , while an admissible ideal  $I$  of  $X$  further satisfies  $\{x\} \in I$  for each  $x \in X$  (see [17]).

A sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  is said to be  $I$ -convergent to  $x \in X$ , if for each  $\epsilon > 0$  the set  $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\}$  belongs to  $I$  (see [16]). A non empty family of sets  $F \subseteq 2^X$  is said to be filter on  $X$  if and only if  $\phi \notin F$ , for  $A, B \in F$  we have  $A \cap B \in F$  and for each  $A \in F$  and  $A \subseteq B$  implies  $B \in F$ . An ideal  $I \subseteq 2^X$  is called non trivial if  $I \neq 2^X$ . A non-trivial ideal  $I \subseteq 2^X$  is called admissible if  $\{\{x\} : x \in X\} \subseteq I$ . A non-trivial ideal is maximal if there cannot exist any non-trivial ideal  $J \neq I$  containing  $I$  as a subset. Further details on ideals of  $2^X$  can be found in [21].

An Orlicz function  $M : [0, \infty) \rightarrow [0, \infty)$  is a continuous, non-decreasing and convex function such that  $M(0) = 0, M(x) > 0$  for  $x > 0$  and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

Lindenstrauss and Tzafiri [23] used the idea of Orlicz function to define the following sequence space

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is known as an Orlicz sequence space. Also  $\ell_M$  is a Banach space with the norm

$$\|(x_k)\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Also, it was shown in [23] that every Orlicz sequence space  $\ell_M$  contains a subspace isomorphic to  $\ell_p (p \geq 1)$ . An Orlicz function  $M$  satisfies  $\Delta_2$ -condition if and only if for any constant  $L > 1$  there exists a constant  $K(L)$  such that  $M(Lu) \leq K(L)M(u)$  for all values of  $u \geq 0$ .

A sequence  $\mathcal{M} = (M_k)$  of Orlicz functions is called a Musielak-Orlicz function see ([24], [29]).

A Musielak-Orlicz function  $(M_k)$  is said to satisfy  $\Delta_2$ -condition if there exist constants  $a, K > 0$  and a sequence  $c = (c_k)_{k=1}^{\infty} \in \ell_+^1$  (the positive cone of  $\ell^1$ ) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all  $k \in \mathbb{N}$  and  $u \in \mathbb{R}_+$  whenever  $M_k(u) \leq a$ .

The notion of difference sequence spaces was introduced by Kızmaz [22], who studied the difference sequence spaces  $l_{\infty}(\Delta)$ ,  $c(\Delta)$  and  $c_0(\Delta)$ . The notion was further generalized by Et and Çolak [11] by introducing the spaces  $\ell_{\infty}(\Delta^n)$ ,  $c(\Delta^n)$  and  $c_0(\Delta^n)$ . Let  $w$  be the space of all complex or real sequences  $x = (x_k)$  and let  $m, n$  be non-negative integers, then for  $Z = \ell_{\infty}, c, c_0$  we have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where  $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$  and  $\Delta_n^0 x_k = x_k$  for all  $k \in \mathbb{N}$ , which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^m (-1)^v \binom{m}{v} x_{k+nv}.$$

Taking  $n = 1$ , we get the spaces which were studied by Et and Çolak [11]. Taking  $m = n = 1$ , we get the spaces which were introduced and studied by Kızmaz [22]. For more details about sequence spaces see ([6], [7], [8], [9], [10], [26], [30], [31], [32], [33], [36], [37]) and for ideal convergence see ([2], [18], [19], [20], [27], [28], [34], [38], [39], [40]). In 2014, Et and Şengül [5] studied the Cesàro-Type sequence spaces of order  $\alpha$ .

Let  $X$  and  $Y$  be two sequence spaces and  $A = (a_{nk})$  be an infinite matrix of real or complex numbers  $a_{nk}$ , where  $n, k \in \mathbb{N}$ . Then we say that  $A$  defines a matrix mapping from  $X$  into  $Y$  if for every sequence  $x = (x_k)_{k=0}^{\infty} \in X$ , the sequence  $Ax = \{A_n(x)\}_{n=0}^{\infty}$ , the  $A$ -transform of  $x$ , is in  $Y$ , where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N}).$$

By  $(X, Y)$ , we denote the class of all matrices  $A$  such that  $A : X \rightarrow Y$ . Thus,  $A \in (X, Y)$  if and only if the series on the right-hand side of above equation converges for each  $n \in \mathbb{N}$  and every  $x \in X$ .

The matrix domain  $X_A$  of an infinite matrix  $A$  in a sequence space  $X$  is defined by

$$X_A = \{x = (x_k) : Ax \in X\}.$$

A sequence space  $E$  is said to be solid (or normal) if  $(x_k) \in E$  implies  $(\gamma_k x_k) \in E$  for all sequences of scalars  $(\gamma_k)$  with  $|\gamma_k| \leq 1$  and for all  $k \in \mathbb{N}$ .

Let  $I$  be an admissible ideal of  $\mathbb{N}$ , let  $p = (p_k)$  be a bounded sequence of positive real numbers and  $A = (a_{nk})$  be an infinite matrix. Let  $\mathcal{M} = (M_k)$  be a Musielak-Orlicz function,  $u = (u_k)$  be a sequence of positive real numbers and  $(X, \|\cdot, \dots, \cdot\|)$  be an  $n$ -normed space. Let  $\alpha \in [0, 1)$  be any real number and  $\Lambda = (\lambda_n)$  be a non-decreasing sequence of positive real numbers such that  $\lambda_{n+1} \leq \lambda_n + 1$ ,  $\lambda_1 = 1$ ,  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Further  $w(n-X)$  denotes the space of all  $X$ -valued sequences. For every  $z_1, \dots, z_{n-1} \in X$ , for each  $\epsilon > 0$  and for some  $\rho > 0$  we define the following sequence spaces:

$$W^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \text{for given } \epsilon > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k - L}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right]^{p_k} \geq \epsilon \right\} \in I, \text{ for some } L \in X \text{ and } s \geq 0 \right\},$$

$$W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \text{for given } \epsilon > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right]^{p_k} \geq \epsilon \right\} \in I, \text{ for some } s \geq 0 \right\}$$

and

$$W_\infty^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \exists K > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right]^{p_k} \geq K \right\} \in I, \text{ for some } s \geq 0 \right\},$$

where  $I_n = [n - \lambda_n + 1, n]$ . Some special cases of the above defined sequence spaces are arises:

If  $m = 0$ , then we obtain the spaces as follows:

$$W^I[\Lambda, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \text{for given } \epsilon > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k x_k - L}{\rho} \right\| \right) \right]^{p_k} \geq \epsilon \right\} \in I, \text{ for some } L \in X \text{ and } s \geq 0 \right\},$$

$$\left. z_1, \dots, z_{n-1} \right\| \Big)^{p_k} \geq \epsilon \Big\} \in I, \text{ for some } L \in X \text{ and } s \geq 0 \Big\},$$

$$W_0^I[\Lambda, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] =$$

$$\left\{ x = (x_k) \in w(n-X) : \text{for given } \epsilon > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \geq \epsilon \right] \in I, \text{ for some } s \geq 0 \right\} \right\}$$

and

$$W_\infty^I[\Lambda, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] =$$

$$\left\{ x = (x_k) \in w(n-X) : \exists K > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \geq K \right] \in I, \text{ for some } s \geq 0 \right\} \right\}.$$

If  $m = n = 1$ , then the above spaces are as follows:

$$W^I[\Lambda, \Delta, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] =$$

$$\left\{ x = (x_k) \in w(n-X) : \text{for given } \epsilon > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta x_k - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \geq \epsilon \right] \in I, \text{ for some } L \in X \text{ and } s \geq 0 \right\} \right\},$$

$$W_0^I[\Lambda, \Delta, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] =$$

$$\left\{ x = (x_k) \in w(n-X) : \text{for given } \epsilon > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \geq \epsilon \right] \in I, \text{ for some } s \geq 0 \right\} \right\}$$

and

$$W_\infty^I[\Lambda, \Delta, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] =$$

$$\left\{ x = (x_k) \in w(n-X) : \exists K > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \geq K \right] \in I, \text{ for some } s \geq 0 \right\} \right\}.$$

If  $s = 0$  and  $\mathcal{M}(x) = x$  for all  $x \in [0, \infty)$ , then we have

$$W^I[\Lambda, \Delta_n^m, A, u, p, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \text{for given } \epsilon > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left( \left\| \frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \geq \epsilon \right\} \in I, \text{ for some } L \in X \text{ and } s \geq 0 \right\},$$

$$W_0^I[\Lambda, \Delta_n^m, A, u, p, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \text{for given } \epsilon > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \geq \epsilon \right\} \in I, \text{ for some } s \geq 0 \right\}$$

and

$$W_\infty^I[\Lambda, \Delta_n^m, A, u, p, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \exists K > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \geq K \right\} \in I, \text{ for some } s \geq 0 \right\}.$$

If  $p = (p_k) = 1$  for all  $k$ , then the above spaces are as follows:

$$W^I[\Lambda, A, \Delta_n^m, \mathcal{M}, u, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \text{for given } \epsilon > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right] \geq \epsilon \right\} \in I, \text{ for some } L \in X \text{ and } s \geq 0 \right\},$$

$$W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \text{for given } \epsilon > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right] \geq \epsilon \right\} \in I, \text{ for some } s \geq 0 \right\}$$

and

$$W_\infty^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \exists K > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right] \geq K \right\} \in I, \text{ for some } s \geq 0 \right\}$$

$$\left. \left. z_{n-1} \right\| \right) \geq K \} \in I, \text{ for some } s \geq 0 \}.$$

If  $A = (C, 1)$ , the Cesàro matrix, then the above spaces are as follows:

$$W^I[\Lambda, \Delta_n^m, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \text{for given } \epsilon > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k - L}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right]^{p_k} \geq \epsilon \right\} \in I, \text{ for some } L \in X \text{ and } s \geq 0 \right\},$$

$$W_0^I[\Lambda, \Delta_n^m, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \text{for given } \epsilon > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right]^{p_k} \geq \epsilon \right\} \in I, \text{ for some } s \geq 0 \right\}$$

and

$$W_\infty^I[\Lambda, \Delta_n^m, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] = \left\{ x = (x_k) \in w(n-X) : \exists K > 0, \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq K \right\} \in I, \text{ for some } s \geq 0 \right\}.$$

By a lacunary sequence  $\theta = (k_r); r = 0, 1, 2, \dots$  where  $k_0 = 0$ , we shall mean an increasing sequence of non-negative integers with  $k_r - k_{r-1} \rightarrow \infty$  as  $r \rightarrow \infty$ . The intervals determined by  $\theta$  will be denoted by  $I_r = (k_{r-1}, k_r]$  and  $h_r = k_r - k_{r-1}$ . We finally arrived, let

$$a_{nk} = \begin{cases} \frac{1}{h_r}, & \text{if } k_{r-1} < k < k_r \\ 0, & \text{otherwise.} \end{cases}$$

Then the above classes of sequences are denoted by  $W^I[\Lambda, \theta, \Delta_n^m, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]$ ,  $W_0^I[\Lambda, \theta, \Delta_n^m, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]$  and  $W_\infty^I[\Lambda, \theta, \Delta_n^m, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]$ .

The following inequality will be used throughout the paper. If  $0 \leq p_k \leq \sup p_k = G$ ,  $D = \max(1, 2^{G-1})$  then

$$(1.1) \quad |a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}$$

for all  $k$  and  $a_k, b_k \in \mathbb{R}$ . Also  $|a|^{p_k} \leq \max(1, |a|^G)$  for all  $a \in \mathbb{R}$ .

The main aim of this paper is to introduce some generalized difference sequence spaces defined by ideal convergence, Musielak-Orlicz function and an infinite matrix of order  $\alpha$ . We have also make an effort to study some inclusion relations and their topological properties.

## 2. Main Results

**Theorem 2.1.** Let  $\mathcal{M} = (M_k)$  be a Musielak-Orlicz function,  $p = (p_k)$  be a bounded sequence of positive real numbers and  $u = (u_k)$  be a sequence of positive real numbers. Then  $W^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$ ,  $W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$  and  $W_\infty^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$  are linear spaces over the real field  $\mathbb{R}$ .

*Proof.* We shall prove the result for the space  $W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$ . Let  $x = (x_k)$  and  $y = (y_k)$  be two elements of  $W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$ . Then there exists  $\rho_1 > 0$  and  $\rho_2 > 0$  and for some  $z_1, \dots, z_{n-1} \in X$  such that

$$A_{\frac{\epsilon}{2}} = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\epsilon}{2} \right\} \in I$$

and

$$B_{\frac{\epsilon}{2}} = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\epsilon}{2} \right\} \in I.$$

Let  $\beta, \gamma \in \mathbb{R}$ . Since  $\|\cdot, \dots, \cdot\|$  is an  $n$ -norm,  $\Delta_n^m$  is linear and the contributing of  $\mathcal{M} = (M_k)$ , the following inequality holds:

$$\begin{aligned} & \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m (\beta x_k + \gamma y_k)}{|\beta| \rho_1 + |\gamma| \rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \leq D \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ \frac{|\beta|}{|\beta| \rho_1 + |\gamma| \rho_2} k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & + D \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ \frac{|\gamma|}{|\beta| \rho_1 + |\gamma| \rho_2} k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \leq DK \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & + DK \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k}, \end{aligned}$$

where  $K = \max \left\{ 1, \frac{|\beta|}{|\beta| \rho_1 + |\gamma| \rho_2}, \frac{|\gamma|}{|\beta| \rho_1 + |\gamma| \rho_2} \right\}$ .

From the above relation, we have

$$\begin{aligned} & \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m (\beta x_k + \gamma y_k)}{|\beta| \rho_1 + |\gamma| \rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \epsilon \right\} \\ & \subseteq \left\{ n \in \mathbb{N} : DK \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\epsilon}{2} \right\} \\ & \cup \left\{ n \in \mathbb{N} : DK \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\epsilon}{2} \right\}. \end{aligned}$$

Since both sets on the R.H.S of above relation are belongs to  $I$ , so the set on the L.H.S of the inclusion relation belongs to  $I$ . Similarly we can prove other cases. This completes the proof of the theorem.  $\square$



**Theorem 2.2.** *Let  $\mathcal{M}' = (M'_k)$  and  $\mathcal{M}'' = (M''_k)$  be two Musielak-Orlicz functions. Then we have  $W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}', u, p, \|\cdot, \dots, \cdot\|] \cap W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}'', u, p, \|\cdot, \dots, \cdot\|] \subseteq W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}' + \mathcal{M}'', u, p, \|\cdot, \dots, \cdot\|]$ .*

*Proof.* Let  $x = (x_k) \in W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}', u, p, \|\cdot, \dots, \cdot\|] \cap W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}'', u, p, \|\cdot, \dots, \cdot\|]$ . Then we get the result by the following inequality:

$$\begin{aligned} & \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} (M'_k + M''_k) \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \leq D \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M'_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \quad + D \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M''_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k}. \end{aligned}$$

Hence,

$$\begin{aligned} & \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} (M'_k + M''_k) \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \epsilon \right\} \\ & \subseteq \left\{ n \in \mathbb{N} : D \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M'_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\epsilon}{2} \right\} \\ & \cup \left\{ n \in \mathbb{N} : D \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M''_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\epsilon}{2} \right\}. \end{aligned}$$

Since both sets on the R.H.S of above relation are belongs to  $I$ , so the set on the L.H.S of the inclusion relation belongs to  $I$ . This completes the proof of the theorem.  $\square$

**Theorem 2.3.** *The inclusions  $Z[\Lambda, \Delta_n^{m-1}, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] \subseteq Z[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$  are strict for  $m \geq 1$ . In general  $Z[\Lambda, \Delta_n^{m-1}, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|] \subseteq Z[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$ , for  $m = 0, 1, 2, \dots$  where  $Z = W^I, W_0^I, W_\infty^I$ .*

*Proof.* We give the proof for  $W_0^I[\Lambda, \Delta_n^{m-1}, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$  only and others can be proved by similar argument. Let  $x = (x_k)$  be any element in the space  $W_0^I[\Lambda, \Delta_n^{m-1}, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$ . Let  $\epsilon > 0$  be given. Then there exists  $\rho > 0$  such that

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^{m-1} x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \epsilon \right\} \in I.$$

Since  $\mathcal{M} = (M_k)$  is non-decreasing and convex for every  $k$ , it follows that

$$\begin{aligned}
 & \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{2\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\
 &= \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^{m-1} x_{k+1} - u_k \Delta_n^{m-1} x_k}{2\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\
 &\leq D \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ \frac{1}{2} k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^{m-1} x_{k+1}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\
 &+ D \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ \frac{1}{2} k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^{m-1} x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\
 &\leq DH \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^{m-1} x_{k+1}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\
 &+ DH \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^{m-1} x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k},
 \end{aligned}$$

where  $H = \max \left\{ 1, \left( \frac{1}{2} \right)^G \right\}$ . Thus we have

$$\begin{aligned}
 & \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{2\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \epsilon \right\} \\
 &\subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^{m-1} x_{k+1}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\epsilon}{2} \right\} \\
 &\cup \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^{m-1} x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\epsilon}{2} \right\}
 \end{aligned}$$

Since both sets in the right hand side of above relation belongs to  $I$ . Therefore, we obtain

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \epsilon \right\} \in I.$$

This inclusion is strict follows from the following example. □

**Example 2.4.** Let  $M_k(x) = x$ , for all  $k \in \mathbb{N}$ ,  $(u_k) = (p_k) = 1$  for all  $k \in \mathbb{N}$ ,  $s = 0$ ,  $\lambda_n^\alpha = 1$ , for  $\alpha \in [0, 1)$  be any real number and  $A = (C, 1)$ , the Cesàro matrix. Now consider a sequence  $x = (x_k) = (k^t)$ . Then for  $n = 1, x = (x_k)$  belongs to  $W_0^I[\Lambda, \Delta_n^m, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$  but does not belongs to  $W_0^I[\Lambda, \Delta_n^{m-1}, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$ , because  $\Delta_n^m x_k = 0$  and  $\Delta_n^{m-1} x_k = (-1)^{m-1} (m-1)!$ .

**Theorem 2.5.** For any two sequences  $p = (p_k)$  and  $q = (q_k)$  of positive real numbers and for any two  $n$ -norms  $\|\cdot, \dots, \cdot\|_1$  and  $\|\cdot, \dots, \cdot\|_2$  on  $X$ , we have the following:

$$Z[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|_1] \cap Z[\Lambda, \Delta_n^m, A, \mathcal{M}, u, q, \|\cdot, \dots, \cdot\|_2] \neq \phi,$$

where  $Z = W^I, W_0^I$  and  $W_\infty^I$ .

*Proof.* Since the zero element belongs to both the classes of sequences, so the intersection is non-empty. □

**Theorem 2.6.** *The sequence spaces  $W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$  and  $W_\infty^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$  are normal as well as monotone.*

*Proof.* We shall prove the theorem for  $W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$ . Let  $x = (x_k) \in W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$  and  $(\gamma_k)$  be a sequence of scalars such that  $|\gamma_k| \leq 1$  for all  $k \in \mathbb{N}$ . Then for given  $\epsilon > 0$ , we have

$$\begin{aligned} & \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m(\gamma_k x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \epsilon \right\} \\ & \subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[ k^{-s} M_k \left( \left\| \frac{u_k \Delta_n^m(x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \epsilon \right\} \in I. \end{aligned}$$

Hence,  $(\gamma_k x_k) \in W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$ . Thus, the space  $W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$  is normal. Therefore,  $W_0^I[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|\cdot, \dots, \cdot\|]$  is monotone also. Similarly, we can prove the theorem for other case. This completes the proof of the theorem.  $\square$

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